

ON THE HEEGAARD FLOER HOMOLOGY FOR BIPARTITE SPATIAL GRAPHS AND ITS PROPERTIES

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ABSTRACT. We discuss a way to define a Heegaard Floer homology for bipartite spatial graphs, and give some properties for this homology. We also give a combinatorial definition of the Euler characteristic of the homology by using the “states” of a given graph projection.

1. INTRODUCTION

The Heegaard Floer homology for knots and links defined by Ozsváth and Szabó [5, 7], and independently Rasmussen [8] has made great impacts on the study of knot theory. In this paper, we attempt to extend their definition to the case of bipartite spatial graphs. Harvey and O’Donnol also announced that they have defined a combinatorial Floer homology for spatial graphs in S^3 .

Our definition in this paper is only for balanced bipartite spatial graphs in a closed oriented 3-manifold. It turns out that, just as the Heegaard Floer homology for links, our definition is a special case of sutured Floer homology defined by Juhász [2]. So it will be helpful to compare with [2] when reading this paper. During the definition, we also need many facts about multi-pointed Heegaard diagrams established in [7].

Heegaard Floer type homology is basically defined on a Heegaard diagram. In Section 2, we define the Heegaard diagram for a bipartite spatial graph G . Then in the case the ground 3-manifold is S^3 , we provide a method for constructing a Heegaard diagram from a connected graph projection of G on S^2 . In Section 3, we define our Heegaard Floer homology for balanced bipartite spatial graphs. In Section 4, some properties of this homology are given. In particular, we provide a relation between the Heegaard Floer homology for a graph and those for some of its subgraphs. In Section 5, we consider the Euler characteristic of the homology in the case the ground 3-manifold is S^3 . Precisely, we define the Euler characteristic combinatorially by using the “states” of the graph projection, which can be regarded as an extension of Kauffman’s definition [3] of the Alexander polynomial for links.

2. HEEGAARD DIAGRAMS

Definition 2.1. A graph G is called a *bipartite graph* if its vertex set V is a disjoint union of two non-empty sets V_1 and V_2 so that there is no edge incident to vertices from the same V_i for $i = 1, 2$. We denote the graph by G_{V_1, V_2} . If $|V_1| = |V_2|$, G is called *balanced*.

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In this paper, we do not consider those graphs with isolated vertices.

The splitting $V = V_1 \amalg V_2$ is not necessarily unique. But when G is connected, it is very easy to see the following lemma.

Lemma 2.2. *If a bipartite graph G is connected, then the choice of the unordered pair V_1 and V_2 is unique.*

Given a compact closed oriented 3-manifold M , we can consider a smooth embedding of a graph $G = (V, E)$ into M , the vertices of V corresponding to points in M and the edges of E corresponding to pairwise disjoint simple arcs in M . We call the isotopy class of such an embedding a spatial graph in M and still denote it by G if no confusion is caused. All manifolds considered below are assumed to be compact and oriented and we work in the smooth category.

Definition 2.3. Suppose G_{V_1, V_2} is a balanced bipartite spatial graph in a closed 3-manifold M . A quartet $(\Sigma, \alpha, \beta, \mathbf{z})$ is called a Heegaard diagram for G_{V_1, V_2} if it satisfies the following conditions.

- (i) Σ is an oriented genus g closed surface, which is called the Heegaard surface. $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ and $\beta = \{\beta_1, \beta_2, \dots, \beta_d\}$ are d_1 -tuple and d_2 -tuple of oriented simple closed curves on Σ respectively, and $\mathbf{z} = \{z_1, z_2, \dots, z_m\}$ is an m -tuple of points in $\Sigma \setminus (\alpha \cup \beta)$, called basepoints. Here $|V_i| = d_i - g + 1$ for $i = 1, 2$ and m is the number of edges of G_{V_1, V_2} .
- (ii) Attaching 2-handles to $\Sigma \times [-1, 0]$ (resp. $\Sigma \times [0, 1]$) along $\alpha \subset \Sigma \times \{-1\}$ (resp. $\beta \subset \Sigma \times \{1\}$), we get a $(d_1 - g + 1)$ -punctured (resp. $(d_2 - g + 1)$ -punctured) handlebody which we call U_α (resp. U_β). We cap off the 2-sphere components of U_α and U_β , except Σ in the case $\Sigma = S^2$, to get handlebodies \overline{U}_α and \overline{U}_β respectively. Then $\overline{U}_\alpha \cup_{\Sigma \times \{0\}} \overline{U}_\beta$ is the 3-manifold M . The orientation of Σ is induced from that of U_α , which in turn is inherited from that of M .
- (iii) For each vertex $v \in V_1$ (resp. $u \in V_2$) whose valency is l (resp. s), there is a smooth embedding $\varphi_v : (\bigsqcup_1^l \mathbb{D}^2, \{1, 2, \dots, l\}) \hookrightarrow (\Sigma \setminus \alpha, \mathbf{z})$ (resp. $\psi_u : (\bigsqcup_1^s \mathbb{D}^2, \{1, 2, \dots, s\}) \hookrightarrow (\Sigma \setminus \beta, \mathbf{z})$). The images of φ_{v_i} for $v_i \in V_1$ (resp. ψ_{u_j} for $u_j \in V_2$) are disjoint from each other except at endpoints. Moreover, $(\bigcup_{v_i \in V_1} \text{Im}(\varphi_{v_i})) \cup (\bigcup_{u_j \in V_2} \text{Im}(\psi_{u_j}))$ is G_{V_1, V_2} . Here we push $\text{Im}(\varphi_{v_i})$ (resp. $\text{Im}(\psi_{u_j})$) slightly into \overline{U}_α (resp. \overline{U}_β).

Any balanced bipartite graph has such Heegaard diagrams. When G_{V_1, V_2} is balanced, namely $|V_1| = |V_2|$, we naturally have $d_1 = d_2$, and we call such a Heegaard diagram a balanced one.

In the case $M \cong S^3$ we extend the idea in [4] and provide an algorithm to construct a Heegaard diagram for a balanced bipartite graph from its graph projection on the 2-sphere S^2 . Consider a graph projection $D \subset S^2$ for a given balanced bipartite spatial graph $G_{V_1, V_2} \subset S^3$. We assume that D is connected.

- (i) Take a tubular neighbourhood of D in S^3 . It is a handlebody and its boundary is the Heegaard surface Σ .
- (ii) For each crossing of D , introduce an α -curve following the rule in Figure 1 (B). The diagram D separates S^2 into several regions. Choose a region and call it β_0 . For each region except β_0 , introduce a β -curve which spans the region as in Figure 1 (A).

- (iii) Choose a vertex $u \in V_2$ with valency l , and introduce $l - 1$ α -curves and l basepoints as in Figure 1 (A). For each vertex $v \in V_2 - \{u\}$, introduce $l - 1$ α -curves, l basepoints and a β -curve which enclose all the basepoints at that vertex. See Figure 1 (C).

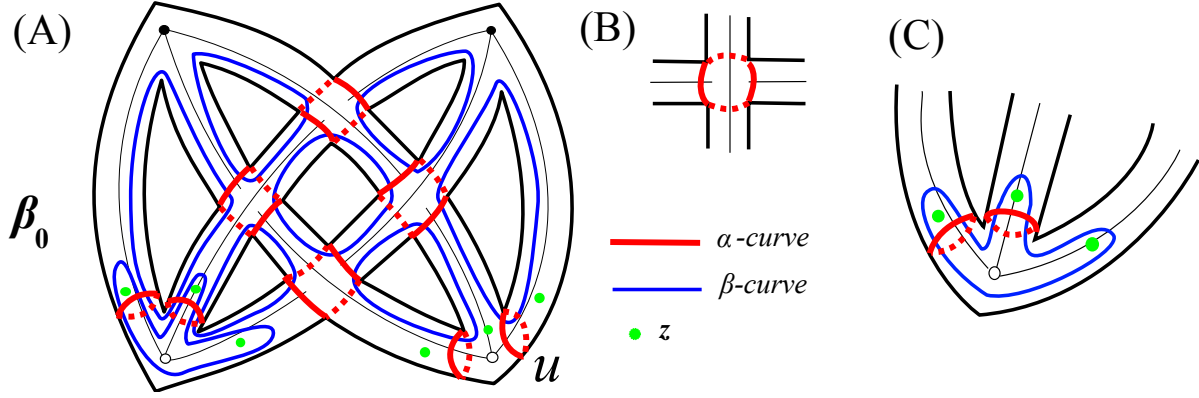


FIGURE 1. The Heegaard diagram associated with a graph projection.

By a standard argument in Morse theory, we have the following theorem. The proof is analogous to that of [2, Proposition 2.15].

Theorem 2.4. *Two Heegaard diagrams $(\Sigma, \alpha, \beta, z)$ and $(\Sigma', \alpha', \beta', z')$ for a given balanced bipartite spatial graph G_{V_1, V_2} can be connected by a finite sequence of the following moves:*

- (i) *Isotopies of α -curves and β -curves which are disjoint from the basepoints;*
- (ii) *Handleslides among α -curves (resp. β -curves) which keep away from the basepoints;*
- (iii) *(De)stabilizations.*

Definition 2.5. For a Heegaard diagram $(\Sigma, \alpha, \beta, z)$, let D_1, D_2, \dots, D_h denote the closures of the components of $\Sigma \setminus (\alpha \cup \beta)$. A *domain* is of the form $D = \sum_{i=1}^h a_i D_i$ with $a_i \in \mathbb{Z}$. D is a *positive domain* if $a_i \geq 0$ for $1 \leq i \leq h$. D is a *periodic domain* if ∂D is a sum of α -curves and β -curves and $n_z(D) = \{0\}$. Here we use $n_p(D)$ to denote the local multiplicity of D at $p \in \Sigma \setminus (\alpha \cup \beta)$ and $n_z(D) := \{n_{z_1}(D), n_{z_2}(D), \dots, n_{z_m}(D)\}$.

Definition 2.6. A Heegaard diagram is said to be *admissible* if every non-trivial periodic domain has both positive and negative local coefficients at the Heegaard surface.

The following Proposition is essentially a corollary of [2, Corollary 3.12]. Here we prove it in a different way.

Proposition 2.7. *If $H_1(M, \mathbb{Z}) = 0$ and the balanced bipartite spatial graph G_{V_1, V_2} is connected, then there is no non-trivial periodic domain on the Heegaard diagram. Therefore any Heegaard diagram of G_{V_1, V_2} is admissible.*

Proof. Suppose the closures of the components of $\Sigma \setminus \alpha$ are $\{A_1, A_2, \dots, A_{d-g+1}\}$, and the closures of the components of $\Sigma \setminus \beta$ are $\{B_1, B_2, \dots, B_{d-g+1}\}$. Then by [7, Sequence (4)]

any periodic domain has the form

$$P = \sum_{i=1}^{d-g+1} (a_i A_i + b_i B_i),$$

for $a_i, b_i \in \mathbb{Z}$. The pieces A_i and B_i correspond to vertices in V_1 and V_2 respectively, and basepoints correspond to edges in E . Any two vertices $v, v' \in V_1$ are connected by a path from the graph. We orient the path from v to v' . Label the elements in V_1, V_2 and E so that when we travel along the path we meet $v = v_1 \in V_1, u_1 \in V_2, v_2 \in V_1, u_2 \in V_2, \dots, v_J \in V_1$ and cross $e_1, e_2, \dots, e_{2J-2}$ with $e_i \in E$. In other words, we can relabel $\{A_1, A_2, \dots, A_{d-g+1}\}, \{B_1, B_2, \dots, B_{d-g+1}\}$ and \mathbf{z} so that $z_{2i-1} \in A_i \cap B_i$ for $1 \leq i \leq J-1$ and $z_{2i} \in B_i \cap A_{i+1}$ for $1 \leq i \leq J-1$. By the condition that $n_{z_{2i-1}}(P) = 0$, we have $a_i + b_i = 0$ for $1 \leq i \leq J-1$, and by the condition that $n_{z_{2i}}(P) = 0$, we have $a_{i+1} + b_i = 0$ for $1 \leq i \leq J-1$. Therefore there exists a constant c so that $a_1 = -b_1 = a_2 = -b_2 = \dots = a_J = c$. By the arbitrariness of v and v' , we finally have $a_i = c$ and $b_i = -c$ for all $1 \leq i \leq d-g+1$. Therefore $P = \sum_{i=1}^{d-g+1} (a_i A_i + b_i B_i) = c(\sum_{i=1}^{d-g+1} A_i - \sum_{i=1}^{d-g+1} B_i) = c(\Sigma - \Sigma) = \emptyset$. \square

In particular when $M = S^3$, the Heegaard diagram constructed after Definition 2.3 is always admissible in the case that G_{V_1, V_2} is connected. But when it is not connected, the Heegaard diagram obtained is usually not admissible.

Later on, we will see that not all Heegaard diagrams can be used in the definition of Heegaard Floer homology. The Heegaard Floer complex is only well defined on an admissible Heegaard diagram. In fact, even though the condition of Proposition 2.7 is not satisfied, we can always make a Heegaard diagram admissible by isotopies of β -curves supported in the complement of \mathbf{z} . The proof can be found in [7, Proposition 3.6] and [2, Proposition 3.15]. Moreover, two admissible Heegaard diagrams can be connected by Heegaard moves so that the intermediate Heegaard diagrams are all admissible. From now on, we only work with admissible Heegaard diagrams.

3. HEEGAARD FLOER COMPLEX

The chain complex defined here should be regarded as an extension of that for link Floer homology, or a special case of sutured Floer homology (See Propersion 4.1). But we give a self-contained definition here, which will be convenient for further discussion.

Most terminologies and notations used here are inherited from [6]. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ be an admissible Heegaard diagram for the balanced bipartite spatial graph G_{V_1, V_2} in the closed 3-manifold M . Define

$$\begin{aligned} \text{Sym}^d(\Sigma) &= \Sigma^{\times d} / S_d, \\ \mathbb{T}_{\boldsymbol{\alpha}} &= (\alpha_1 \times \alpha_2 \times \dots \times \alpha_d) / S_d \\ \text{and } \mathbb{T}_{\boldsymbol{\beta}} &= (\beta_1 \times \beta_2 \times \dots \times \beta_d) / S_d, \end{aligned}$$

where S_d is the symmetric group with d letters.

Following the method in [6], we can show that $\text{Sym}^d(\Sigma)$ is a smooth manifold, and that a complex structure j on Σ naturally endows $\text{Sym}^d(\Sigma)$ an almost complex structure $\text{Sym}^d(j)$, with respect to which $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$ are totally real submanifolds of $\text{Sym}^d(\Sigma)$.

We can suppose that \mathbb{T}_α and \mathbb{T}_β are in general position, which means that they intersect transversely. The orientation of Σ gives a product orientation for $\text{Sym}^d(\Sigma)$, which we use throughout the paper.

Definition 3.1. Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, a *Whitney disk* connecting x to y is a continuous map $u : \mathbb{D} \rightarrow \text{Sym}^d(\Sigma)$ so that $u(-i) = x$, $u(i) = y$, $u(e_1) \subset \mathbb{T}_\alpha$ and $u(e_2) \subset \mathbb{T}_\beta$. Here \mathbb{D} is the unit disk, $e_1 = \{z \in \partial(\mathbb{D}) | \text{Re}(z) \geq 0\}$ and $e_2 = \{z \in \partial(\mathbb{D}) | \text{Re}(z) \leq 0\}$. Let $\pi(x, y)$ be the set of homotopy classes of Whitney disks connecting x to y .

Given a $\phi \in \pi(x, y)$, let $D(\phi) := \sum_{i=1}^h n_{p_i}(\phi) D_i$ be the associated domain, where p_i is an arbitrary point in $\text{int}(D_i)$ and $n_{p_i}(\phi)$ is the algebraic intersection number $\phi^{-1}(\{p_i\} \times \text{Sym}^{d-1}(\Sigma))$. Let $\mathcal{M}(\phi)$ be the moduli space of pseudo-holomorphic representatives of $\phi \in \pi(x, y)$ and $\widehat{\mathcal{M}}(\phi) := \mathcal{M}(\phi)/\mathbb{R}$ be the unparametrized moduli space. Also, let $\mu(\phi)$ denote the Maslov index of ϕ . When $\mu(\phi) = 1$ the space $\widehat{\mathcal{M}}(\phi)$ is a compact zero-dimensional manifold ([2, Corollary 6.4] and [6, Theorem 3.18]).

Definition 3.2. Let $\text{CFG}(\Sigma, \alpha, \beta, z)$ be the vector space over $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$ generated by points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Also define a linear map

$$\begin{aligned} \partial : \text{CFG}(\Sigma, \alpha, \beta, z) &\rightarrow \text{CFG}(\Sigma, \alpha, \beta, z) \\ x &\mapsto \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x, y), \mu(\phi)=1, n_z(\phi)=\{0\}} \# \widehat{\mathcal{M}}(\phi) \cdot y, \end{aligned}$$

where $\# \widehat{\mathcal{M}}(\phi)$ is the number of elements in $\widehat{\mathcal{M}}(\phi)$ modulo 2. That the right-hand side has only finitely many non-trivial terms follows from the admissibility of the Heegaard diagram.

Theorem 3.3. (i) $(\text{CFG}(\Sigma, \alpha, \beta, z), \partial)$ is a chain complex. Namely $\partial^2 = 0$.
(ii) The homology of the chain complex above is a topological invariant of G_{V_1, V_2} , denoted by $\text{HFG}(M, G_{V_1, V_2})$

Proof. (i) This is analogous to [6, Theorem 4.1]. (ii) With Theorem 2.4 in hand, the proof follows from [6, Section 7~11]. \square

Remark 3.4. If $(\Sigma, \alpha, \beta, z)$ is a Heegaard diagram for G_{V_1, V_2} , then $(-\Sigma, \beta, \alpha, z)$ is a Heegaard diagram for G_{V_2, V_1} . A canonical chain isomorphism between $\text{CFG}(\Sigma, \alpha, \beta, z)$ and $\text{CFG}(-\Sigma, \beta, \alpha, z)$ exists. When G is connected, the choice of V_1 and V_2 is unique. So the homology only depends on the topology type of G . When G is not connected, the splitting $V = V_1 \amalg V_2$ is not unique and the homology depends on the splitting.

3.1. Relative gradings. We define two relative gradings for the complex above.

Define the set

$$S := \{(x_1, x_2, \dots, x_d) | x_i \in \alpha_i \cap \beta_{\sigma(i)} \text{ for some } \sigma \in S_d\}.$$

Then it is easy to see that there is a canonical identification between $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and S . We do not distinguish these two sets sometimes. Given $x, y \in S$, choose a multi-path $a \subset \bigcup_{i=1}^d \alpha_i$ connecting x to y , and a multi-path $b \subset \bigcup_{i=1}^d \beta_i$ connecting y to x . Then $a + b$ is a one-cycle in Σ . The homology class $[a + b] \in H_1(M \setminus \nu(G), \mathbb{Z})$ does not depend on the choice of a and b , where $\nu(G)$ is the interior of a regular neighbourhood of $G \subset M$. This is because for different a' and b' , the difference $(a + b) - (a' + b')$ is a union of some α -curves and β -curves.

Definition 3.5. Given $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, choose a and b as above. Then we define a relative grading $A : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rightarrow H_1(M \setminus \nu(G), \mathbb{Z})$ by the formula

$$A(x) - A(y) = [a + b] \in H_1(M \setminus \nu(G), \mathbb{Z}).$$

Lemma 3.6. *If $A(x) - A(y) \neq 0$, then the subset of elements in $\pi_2(x, y)$ with $n_z(\phi) = \{0\}$ is empty.*

Proof. If there exists a $\phi \in \pi(x, y)$ with $n_z = \{0\}$, then the associated domain $D(\phi)$ is an empty domain connecting x to y . Then $A(x) - A(y) = \partial D(\phi) = 0 \in H_1(M \setminus \nu(G), \mathbb{Z})$. \square

The other one is a relative $\mathbb{Z}/2\mathbb{Z}$ -grading on $\text{CFG}(\Sigma, \alpha, \beta, z)$. As we said in Definition 2.3, the orientation of Σ is inherited from that of M . We orient the α -curves $\alpha_1, \alpha_2, \dots, \alpha_d$ and the β -curves $\beta_1, \beta_2, \dots, \beta_d$ arbitrarily. Then $\text{Sym}^d(\Sigma)$, \mathbb{T}_α and \mathbb{T}_β have the orientations which are induced from the product orientations of $\Sigma^{\times d}$, $\alpha_1 \times \alpha_2 \times \dots \times \alpha_d$ and $\beta_1 \times \beta_2 \times \dots \times \beta_d$, respectively. For a generator $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the sign of the intersection gives its grading in $\mathbb{Z}/2\mathbb{Z} = \{+1, -1\}$, and we denote it by $\text{sign}(x)$.

Identify $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $(x_1, x_2, \dots, x_d) \in S$, where $x_i \in \alpha_1 \cap \beta_{\sigma(i)}$ for some $\sigma \in S_d$. Denote the sign of the intersection point x_i in Σ by $\text{sign}(x_i)$. Then we have the relation $\text{sign}(x) = (-1)^{d(d-1)/2} \text{sign}(\sigma) \prod_{i=1}^d \text{sign}(x_i)$ (see [1, Lemma 2.12]).

We see that the differential in $\text{CFG}(\Sigma, \alpha, \beta, z)$ preserves the grading A and changes the relative $\mathbb{Z}/2\mathbb{Z}$ -grading. So we have the following splitting.

Theorem 3.7. $(\text{CFG}(\Sigma, \alpha, \beta, z), \partial)$ splits along $H_1(M \setminus \nu(G), \mathbb{Z})$. Namely

$$\text{CFG}(\Sigma, \alpha, \beta, z) = \bigoplus_{i \in H_1(M \setminus \nu(G), \mathbb{Z})} \text{CFG}(\Sigma, \alpha, \beta, z, i).$$

We have

$$\text{HFG}_j(M, G_{V_1, V_2}) = \bigoplus_{i \in H_1(S^3 \setminus \nu(G), \mathbb{Z})} \text{HFG}_j(M, G_{V_1, V_2}, i),$$

for $j = +1, -1$.

Definition 3.8. The Euler characteristic for $\text{CFG}(\Sigma, \alpha, \beta, z)$ is

$$\chi(\text{CFG}(\Sigma, \alpha, \beta, z)) := \sum_{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \text{sign}(x) \cdot A(x).$$

4. PROPERTIES

Proposition 4.1. For the balanced bipartite spatial graph G_{V_1, V_2} ,

$$\text{HFG}(M^3, G_{V_1, V_2}) \cong \text{SFH}(M^3 \setminus \nu(G), \gamma),$$

where $\text{SFH}(M^3 \setminus \nu(G), \gamma)$ is the sutured Floer homology with the meridian annuli of all edges as the sutures.

Proof. The proof follows directly from the construction of these two homologies. \square

Proposition 4.2. For the balanced bipartite spatial graph G_{V_1, V_2} , if G_{V_1, V_2} has two vertices $v \in V_1$ and $u \in V_2$ with valencies $d(v) = 1$ and $d(u) > 1$ which are incident to a common edge $e \in E$, then $\text{HFG}(M, G_{V_1, V_2}) = 0$.

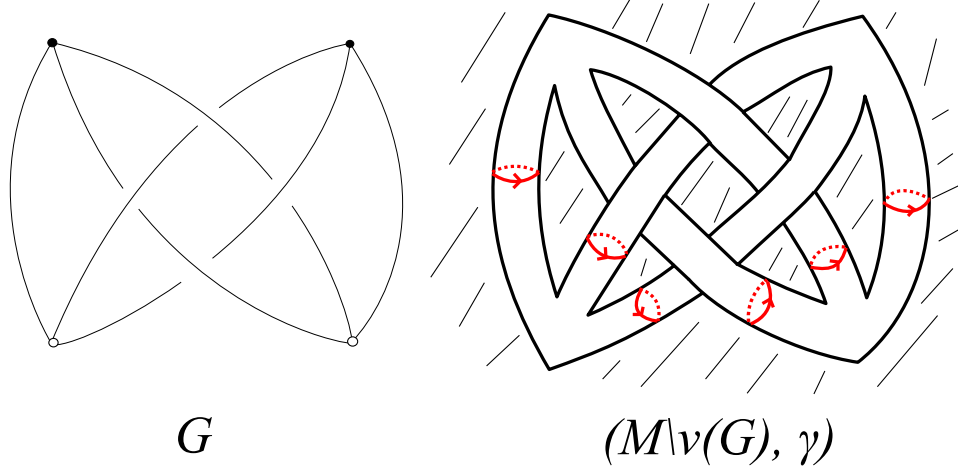


FIGURE 2.

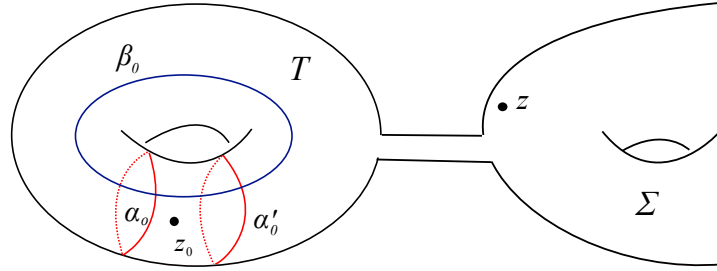


FIGURE 3.

Proof. Removing the edge e and the vertex v from G_{V_1, V_2} , we get a new graph G' , which is not balanced anymore. But we can still consider its Heegaard diagram

$$H' = (\Sigma, \alpha' = \{\alpha_1, \dots, \alpha_{d-1}\}, \beta' = \{\beta_1, \dots, \beta_d\}, \mathbf{z}' = \{z_1, \dots, z_{m-1}\}),$$

where $d - g + 1 = |V_2|$ and $m = |E|$, the number of edges in G_{V_1, V_2} . By applying [2, Proposition 3.15], we can show that H' is isotopic to an admissible Heegaard diagram. In fact, the essential condition needed in the proof is that elements of α' (resp. β') are linearly independent in $H_1(\Sigma \setminus \mathbf{z}', \mathbb{Z})$.

Therefore we can assume that H' itself is admissible. Suppose $z \in \mathbf{z}$ is a basepoint corresponding to an edge incident to u in G' . Then

$$\begin{aligned} H &= H' \sharp (T, \{\alpha_0, \alpha'_0\}, \{\beta_0\}, z_0) \\ &= (\Sigma \sharp T, \alpha = \alpha' \cup \{\alpha_0, \alpha'_0\}, \beta = \beta' \cup \{\beta_0\}, \mathbf{z} = \mathbf{z}' \cup \{z_0\}), \end{aligned}$$

where the connected sum takes place near z , is a Heegaard diagram for G_{V_1, V_2} . A periodic domain P for H is disjoint from z and z_0 , so it must be supported in Σ . This means that P is a periodic domain for H' , having both positive and negative local multiplicities. Therefore H is admissible as well. From H we see that $\mathbb{T}_\alpha \cap \mathbb{T}_\beta = \emptyset$, so $\text{HFG}(M, G_{V_1, V_2}) = 0$. \square

Proposition 4.3. *Suppose G_{V_1, V_2} is a balanced bipartite spatial graph in the integral homology 3-sphere M , and I_{V_1, V_2} is a subgraph of G which has the same component*

number as G_{V_1, V_2} . Then there is a \mathbb{Z}^l filtration so that $\text{HFG}(M, G_{V_1, V_2})$ is the associated graded homology of $\text{HFG}(M, I_{V_1, V_2})$ with respect to this filtration. Here l is the difference between the number of edges for G_{V_1, V_2} and that for I_{V_1, V_2} .

Proof. We only need to consider the case that I is obtained from G by removing a non-separating edge e . Suppose now that $H_G = (\Sigma, \alpha, \beta, \mathbf{z})$ is a Heegaard diagram of G . Then $H_I = (\Sigma, \alpha, \beta, \mathbf{z} \setminus \{z_e\})$ is a Heegaard diagram for I , where z_e corresponds to the edge e . By using the method in the proof of [2, Proposition 3.15], we can make H_G admissible by isotopy moves of β -curves. When we apply the method, we can only use the basepoints in $\mathbf{z} \setminus \{z_e\}$, while keeping the basepoint z_e away from the winding regions. In this way H_I becomes admissible simultaneously. We still use H_G and H_I to denote the admissible Heegaard diagrams after the isotopy moves.

Now $\text{CFG}(H_G)$ and $\text{CFG}(H_I)$ have the same generating set $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. By definition, given a generator $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the differential of $\text{CFG}(H_I)$ is

$$\begin{aligned} \partial^I(x) &= \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x, y), \mu(\phi)=1, \\ n_{\mathbf{z} \setminus \{z_e\}}(\phi)=\{0\}}} \sharp \widehat{\mathcal{M}}(\phi) \cdot y \\ &= \sum_{i=0}^{\infty} \partial_i(y), \end{aligned}$$

where $\partial_i(x) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x, y), \mu(\phi)=1, \\ n_{\mathbf{z} \setminus \{z_e\}}(\phi)=\{0\}, n_{z_e}(\phi)=i}} \sharp \widehat{\mathcal{M}}(\phi) \cdot y$. Here we use the fact that if $\phi \in \pi_2(x, y)$ has a pseudo-holomorphic representative, then $n_p(\phi) \geq 0$ for any $p \in \Sigma \setminus \alpha \cup \beta$. Moreover, ∂_0 coincides with the differential ∂^G in $\text{CFG}(H_G)$.

Since M is an integral homology 3-sphere, we have $H_1(M \setminus \nu(G), \mathbb{Z}) = \mathbb{Z}^{g(G)} = \mathbb{Z}^{g(I)} \oplus \mathbb{Z}\langle m_e \rangle$, where $\mathbb{Z}\langle m_e \rangle$ is a summand generated by the meridian m_e of e whose orientation is induced from that of Σ . (The boundary of a disk neighborhood of z_e on Σ is m_e .) Here $g(G)$ is the summation of the genera of all components of $\overline{\nu(G)}$. For $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, their grading difference in $\text{CFG}(H_G)$ is

$$A^G(x) - A^G(y) = [a + b] \in H_1(M \setminus \nu(G), \mathbb{Z}) = \mathbb{Z}^{g(I)} \oplus \mathbb{Z}\langle m_e \rangle,$$

where a and b are the multi-paths in Σ connecting x to y along curves of α and y to x along curves of β , respectively. The grading difference in $\text{CFG}(H_I)$ is

$$A^I(x) - A^I(y) = [a + b] \in \mathbb{Z}^{g(I)}.$$

In fact, $A^I(x) - A^I(y)$ is the projection of $A^G(x) - A^G(y)$ onto $\mathbb{Z}^{g(I)}$. We define a new grading $B : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rightarrow \mathbb{Z}$ on $\text{CFG}(H_I)$ by requiring that $B(x) - B(y)$ is the projection of $A^G(x) - A^G(y)$ onto $\mathbb{Z}\langle m_e \rangle$. Then if $\partial_i(x) \neq 0$, $B(x) - B(\partial_i(x)) = i$ for $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $i \geq 0$. With respect to the grading B , the associated graded homology of $(\text{CFG}(H_I), \partial_I)$ is $\text{HFG}(M, G_{V_1, V_2})$. □

5. EULER CHARACTERISTIC

Friedl, Juhász and Rasmussen in [1] defined a torsion invariant for sutured manifolds and showed that it is the Euler characteristic of the sutured Floer homology. We see from Proposition 4.1 that the homology we defined in this paper is a special case of sutured Floer homology. In fact, in our case the torsion invariant defined by them is

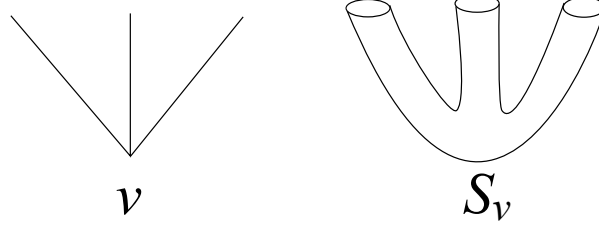


FIGURE 4.

just the maximal abelian torsion of the pair $(M \setminus \nu(G), S_{V_2})$, where $S_{V_2} = \coprod_{v \in V_2} S_v$ with S_v the neighbourhood of v in $\partial(M \setminus \nu(G))$ as shown in Figure 4. We give a combinatorial interpretation of this torsion in the case that $M = S^3$. Precisely, we show how to define it from a graph projection on the 2-sphere S^2 . The construction can be regarded as an extension of Kauffman's construction of the Alexander polynomial for links in his book [3].

Suppose G_{V_1, V_2} is a balanced bipartite spatial graph in S^3 , and D is a connected graph diagram of G_{V_1, V_2} on S^2 . Choosing a vertex $u \in V_2$, we define an element $\tau_u^D \in \mathbb{Z}H_1(S^3 \setminus \nu(G), \mathbb{Z})$ from D and prove that it coincides with the maximal abelian torsion of the pair $(S^3 \setminus \nu(G), S_{V_2})$. We complete the definition by the following steps.

- (i) For those vertices $v \in V_2 - \{u\}$, changing the local neighbourhood of $v \subset S^2$ as in Figure 5, we get a new graph D' from D . Choose a vertex c_v from the newly created vertices. We call the remaining $d(v) - 1$ vertices the *crossings around* v , where $d(v)$ is the valency of v .
- (ii) The graph D' separates S^2 into several regions. Mark the regions adjacent to u by stars $*$. Consider the crossings in D' , which are either double points inherited from D or the crossings created in Step (i). We label them by $\{c_1, c_2, \dots, c_n\}$. We see that there are n regions R_1, R_2, \dots, R_n which are not marked by $*$. See Figure 6.
- (iii) Each double point is locally adjacent to four regions, and each crossing created in Step (i) is locally adjacent to three regions. A *state* of D is a bijection $s : \{c_1, c_2, \dots, c_n\} \rightarrow \{R_1, R_2, \dots, R_n\}$ so that each crossing is mapped to a region adjacent to it. Red dots in Figure 6 shows a state for the given diagram.
- (iv) For each state s , define $\text{sign}(s)$ to be the sign of s as a permutation. Orient each edge so that it directs from a vertex in V_2 to a vertex in V_1 , and assign a variable t_i to each edge e_i . Let

$$m(s) := \prod_{i=1}^n \Delta_i^{s(c_i)},$$

where $\Delta_i^{s(c_i)}$ is defined by the rule in Figure 7.

- (v) We define the Laurent polynomial

$$\tau_u^D(t_1, t_2, \dots, t_m) := \sum_s \text{sign}(s) m(s).$$

We choose the presentation $H_1(S^3 \setminus \nu(G), \mathbb{Z}) \cong \langle t_1, t_2, \dots, t_m | r_v, v \in V; t_i t_j = t_j t_i \text{ for } 1 \leq i, j \leq m \rangle$, where $r_v = \prod_{\lambda \in \Lambda} t_\lambda$ with $\{e_\lambda\}_{\lambda \in \Lambda}$ the edges incident to v . Then $\tau_u^D(t_1, t_2, \dots, t_m)$ is regarded as an element of $\mathbb{Z}H_1(S^3 \setminus \nu(G), \mathbb{Z})$.

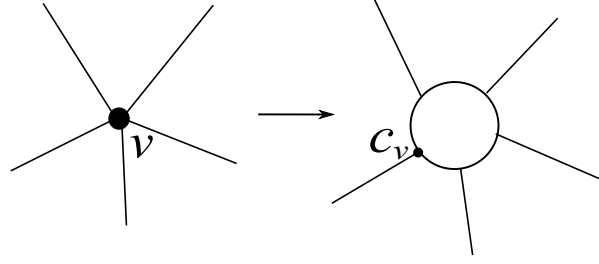


FIGURE 5.

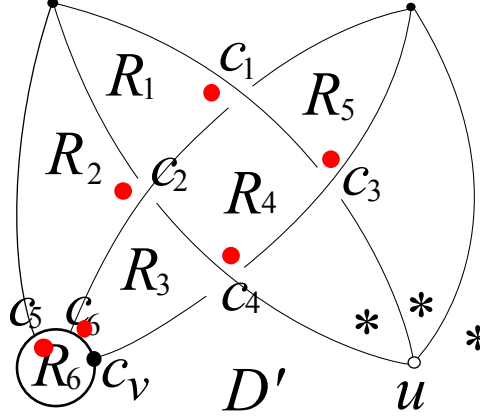
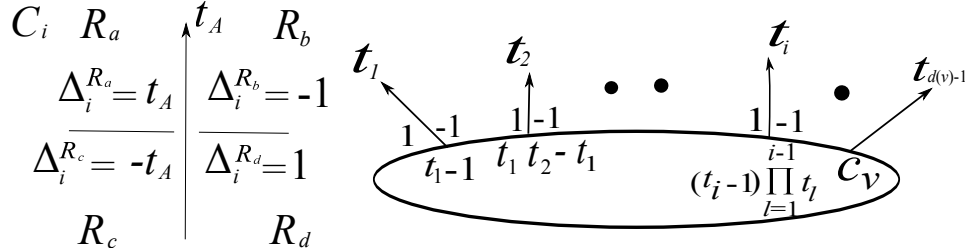
FIGURE 6. $s : (c_1, c_2, \dots, c_6) \rightarrow (R_1, R_2, R_5, R_4, R_6, R_3)$.

FIGURE 7.

Theorem 5.1. Up to an overall multiplication of an element in $H_1(S^3 \setminus \nu(G), \mathbb{Z})$, the polynomial $\tau_u^D(t_1, t_2, \dots, t_m)$ coincides with the maximal abelian torsion of the pair $(S^3 \setminus \nu(G), S_{V_2})$, and therefore the Euler characteristic of $\text{HFG}(S^3, G)$.

Example 5.2. For the graph projection in Figure 8, we see there are five states. State s_1 sends $(c_1, c_2, c_3, c_4, c_5)$ to $(R_2, R_3, R_4, R_1, R_5)$, so it has $\text{sign}(s_1) = -1$. Similarly, we see that $\text{sign}(s_2) = \text{sign}(s_5) = 1$ and $\text{sign}(s_3) = \text{sign}(s_4) = -1$. Following the rules in Figure 7, we calculate $m(s_i)$ for $1 \leq i \leq 5$. We get $m(s_1) = t_2^3$, $m(s_2) = 1$, $m(s_3) = t_2$, $m(s_4) = t_3$ and $m(s_5) = t_2^2$. Therefore $\tau(t_1, t_2, t_3) = -t_2^3 + 1 - t_2 - t_3 + t_2^2$.

Proof of Theorem 5.1. In Section 2, we showed how to construct a Heegaard diagram H_D from a connected graph diagram D of G_{V_1, V_2} . We choose the right-handed orientation for the Heegaard surface of H_D , orient the β -curves counterclockwisly viewed from the

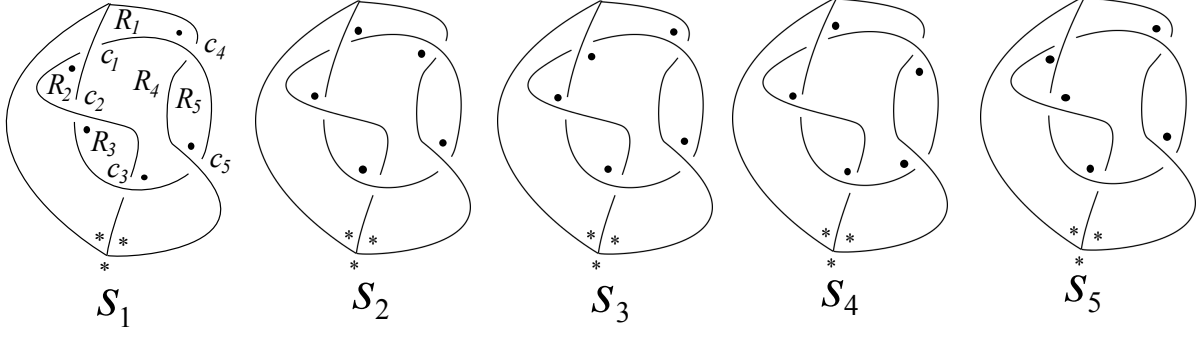


FIGURE 8. There are five states for this projection.

front, and orient the α -curves at double points clockwise and the α -curves around vertices in V_2 as in Figure 10 case (ii), case (iii).

From the Morse function viewpoint, the Heegaard diagram H_D naturally gives a relative handle decomposition of $S^3 \setminus \nu(G)$ built on $S_{V_2} \times I$. We first attach 1-handles to $S_{V_2} \times I$ with belt circles $\beta_1, \beta_2, \dots, \beta_d$, and then attach 2-handles with attaching circles $\alpha_1, \alpha_2, \dots, \alpha_d$. Then the relative chain complex $C_1(S^3 \setminus \nu(G), S_{V_2}, \mathbb{Z})$ is freely generated by $\beta_1^*, \beta_2^*, \dots, \beta_d^*$, where β_i^* is dual to β_i , and $C_2(S^3 \setminus \nu(G), S_{V_2}, \mathbb{Z})$ is freely generated by $\alpha_1, \alpha_2, \dots, \alpha_d$.

The pair $(S^3 \setminus \nu(G), S_{V_2})$ is homotopy equivalent to a 2-complex satisfying the conditions in [1, Lemma 3.6]. Therefore by the lemma the maximal abelian torsion of $(S^3 \setminus \nu(G), S_{V_2})$, which is the Euler characteristic of $\text{HFG}(S^3, G)$, coincides with $\pm \det(A)$, where $A = ([\frac{\partial \alpha_i}{\partial \beta_j}])_{i,j=1}^d$. Here $\frac{\partial \alpha_i}{\partial \beta_j}$ is calculated by the Fox calculus and $[\frac{\partial \alpha_i}{\partial \beta_j}]$ is its image in $\mathbb{Z}H_1(S^3 \setminus \nu(G), \mathbb{Z})$.

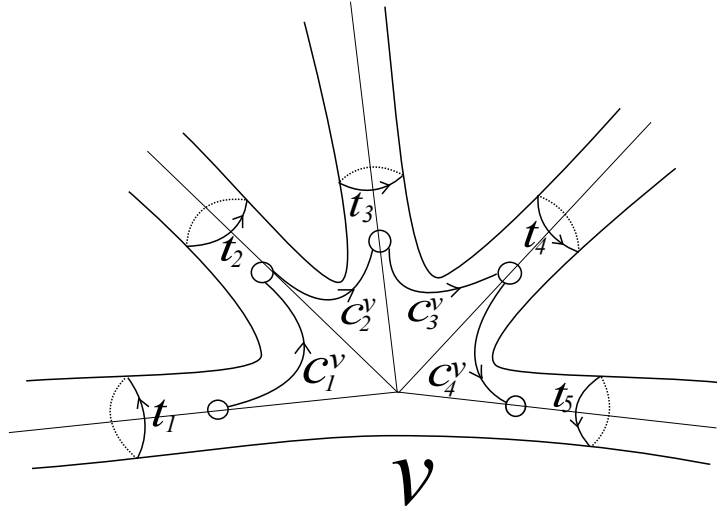


FIGURE 9.

More precisely, as in Figure 9, define arcs $c_1^v, c_2^v, \dots, c_{d(v)-1}^v$ for each vertex $v \in V_2$ with valency $d(v)$. To write down the word α_i , we simply traverse α_i and record its

intersections with the curves β_j and the arcs c_k^v for all $j = 1, 2, \dots, d$, $k = 1, 2, \dots, d(v) - 1$ and $v \in V_2$.

Now we calculate $[\frac{\partial \alpha_i}{\partial \beta_j}]$. We orient the graph D so that each edge directs from a vertex in V_2 to a vertex in V_1 . We choose the presentation $H_1(S^3 \setminus \nu(G), \mathbb{Z}) \cong \langle t_1, t_2, \dots, t_m | r_v, v \in V; t_i t_j = t_j t_i \text{ for } 1 \leq i, j \leq m \rangle$, where $r_v = \prod_{\lambda \in \Lambda} t_\lambda$ with $\{e_\lambda\}_{\lambda \in \Lambda}$ the edges connecting to v . The generator t_i is represented by the loop in Figure 9.

Depending on the position of α_i , we have three cases to consider.

Case (i): α_i corresponds to a double point (see Figure 10 case (i)). In this case, $\alpha_i = \beta_a \beta_b^{-1} \beta_d \beta_c^{-1}$. The indices a, b, c, d are not necessarily different and some of the curves among $\beta_a, \beta_b, \beta_c$ and β_d may not exist at all, but at this moment we treat them as different indices. We see that

$$\begin{aligned} \frac{\partial \alpha_i}{\partial \beta_a} &= 1, & \frac{\partial \alpha_i}{\partial \beta_b} &= -\beta_a \beta_b^{-1}, \\ \frac{\partial \alpha_i}{\partial \beta_c} &= -\beta_a \beta_b^{-1} \beta_d \beta_c^{-1}, & \frac{\partial \alpha_i}{\partial \beta_d} &= \beta_a \beta_b^{-1}. \end{aligned}$$

We have the relations $[\beta_b] = [\beta_a] t_A$ and $[\beta_d] = [\beta_c] t_A$. Therefore

$$[\frac{\partial \alpha_i}{\partial \beta_a}] = 1, [\frac{\partial \alpha_i}{\partial \beta_b}] = -t_A^{-1}, [\frac{\partial \alpha_i}{\partial \beta_c}] = -1, [\frac{\partial \alpha_i}{\partial \beta_d}] = t_A^{-1},$$

and $[\frac{\partial \alpha_i}{\partial \beta_j}] = 0$ for those $j \neq a, b, c, d$. Up to an overall multiplication of an element in $H_1(S^3 \setminus \nu(G), \mathbb{Z})$, the determinant $\det(A)$ does not change if we let

$$[\frac{\partial \alpha_i}{\partial \beta_a}] = t_A, [\frac{\partial \alpha_i}{\partial \beta_b}] = -1, [\frac{\partial \alpha_i}{\partial \beta_c}] = -t_A, [\frac{\partial \alpha_i}{\partial \beta_d}] = 1,$$

and $[\frac{\partial \alpha_i}{\partial \beta_j}] = 0$ for those $j \neq a, b, c, d$.

Case (ii): α_i is around the vertex $u \in V_2$ (see Figure 10 case (ii)). In this case, $\alpha_i = c_k^u \beta_{j_0}$ for some $1 \leq k \leq d(u) - 1$ and $1 \leq j_0 \leq d$. Therefore $[\frac{\partial \alpha_i}{\partial \beta_j}] = [c_k^u]$ if $j = j_0$ and otherwise zero. We let I_u and J_u denote the indices of α -curves and β -curves, respectively. Then up to an overall multiplication of an element in $H_1(S^3 \setminus \nu(G), \mathbb{Z})$, the determinant $\det(A)$ does not change if we let $A = ([\frac{\partial \alpha_i}{\partial \beta_j}])_{i,j=1, i \notin I_u, j \notin J_u}^d$.

Case (iii): α_i is around a vertex $v \in V_2 - \{u\}$ (see Figure 10 case (iii)). In this case $\alpha_i = \beta_a^{-1} \beta_b \beta_{j_0}^{-1} (c_{k-1}^v)^{-1} \beta_{j_0}$ for some $1 \leq k \leq d(v) - 1$ and $1 \leq a, b, j_0 \leq d$, where we define c_0^v to be one. We have

$$\begin{aligned} \frac{\partial \alpha_i}{\partial \beta_a} &= -\beta_a^{-1}, \\ \frac{\partial \alpha_i}{\partial \beta_b} &= \beta_a^{-1}, \\ \frac{\partial \alpha_i}{\partial \beta_{j_0}} &= \beta_a^{-1} \beta_b \beta_{j_0}^{-1} (c_{k-1}^v (c_k^v)^{-1} - 1). \end{aligned}$$

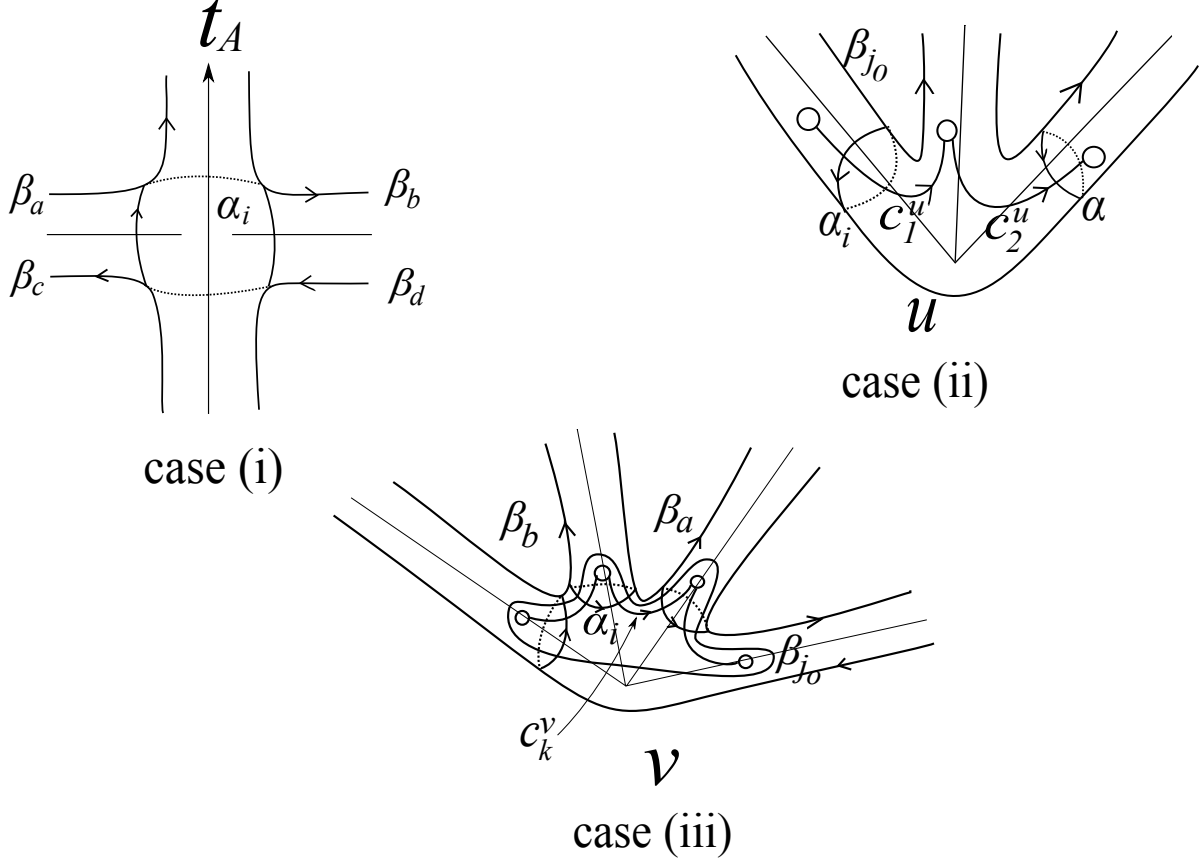


FIGURE 10.

We have the relations $[\beta_a] = [\beta_b]t_k$, $[\beta_a] = [\beta_c]t_1t_2 \cdots t_k$ and $[c_k^v] = t_1^{-1}t_2^{-1} \cdots t_k^{-1}$. Therefore

$$\begin{aligned} \left[\frac{\partial \alpha_i}{\partial \beta_a}\right] &= -[\beta_c]^{-1}t_1^{-1}t_2^{-1} \cdots t_k^{-1}, \\ \left[\frac{\partial \alpha_i}{\partial \beta_b}\right] &= [\beta_c]^{-1}t_1^{-1}t_2^{-1} \cdots t_k^{-1}, \\ \left[\frac{\partial \alpha_i}{\partial \beta_{j_0}}\right] &= [\beta_{j_0}]^{-1}(1 - t_k), \end{aligned}$$

and $\left[\frac{\partial \alpha_i}{\partial \beta_j}\right] = 0$ if $j \neq a, b, j_0$. When we calculate the determinant of the matrix A , up to an overall multiplication of an element in $H_1(S^3 \setminus \nu(G), \mathbb{Z})$, we can replace the entries so that

$$\left[\frac{\partial \alpha_i}{\partial \beta_a}\right] = -1, \left[\frac{\partial \alpha_i}{\partial \beta_b}\right] = 1, \text{ and } \left[\frac{\partial \alpha_i}{\partial \beta_{j_0}}\right] = (t_k - 1)t_1t_2 \cdots t_{k-1}.$$

It is easy to see now that $\det(A) = \tau_u^D$.

□

REFERENCES

- [1] S. FRIEDL, A. JUHÁSZ, AND J. RASMUSSEN, *The decategorification of sutured Floer homology*, J. Topol., 4 (2011), pp. 431–478.
- [2] A. JUHÁSZ, *Holomorphic discs and sutured manifolds*, Algebr. Geom. Topol., 6 (2006), pp. 1429–1457.
- [3] L. H. KAUFFMAN, *Formal knot theory*, vol. 30 of Mathematical Notes, Princeton University Press, Princeton, NJ, 1983.
- [4] P. OZSVÁTH AND Z. SZABÓ, *Heegaard Floer homology and alternating knots*, Geom. Topol., 7 (2003), pp. 225–254 (electronic).
- [5] ———, *Holomorphic disks and knot invariants*, Adv. Math., 186 (2004), pp. 58–116.
- [6] ———, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2), 159 (2004), pp. 1027–1158.
- [7] ———, *Holomorphic disks, link invariants and the multi-variable Alexander polynomial*, Algebr. Geom. Topol., 8 (2008), pp. 615–692.
- [8] J. RASMUSSEN, *Floer homology and knot complements*, Ph. D. thesis, Harvard University, (2003).

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